1. (a) Find the limit of the sequence $\{x_n\}$ as $n \to \infty$, where

$$x_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}.$$

(b) Let g be a non-vanishing continuous real-valued function on $[0, \infty)$ such that $g(x+y) = g(x)g(y)$ for all $x, y \geq 0$. Prove that there exists a real number a such that $g(x) \equiv e^{ax}$.

[5+7]=12

2. (a) Let f be a real-valued function defined as follows:

$$f(x,y) = \begin{cases} \frac{e^{x^2y^2}-1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous everywhere? Justify your answer.

(b) Suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ is a differentiable function such that $f(x) > f(0)$ and $f(x) > f(1)$ for some $x \in (0,1)$. Prove that there exists a point $x_0 \in (0,1)$ such that $f'(x_0) = 0$.

[6+6]=12

3. Let P be an $n \times n$ non-singular matrix such that $I + P + P^2 + \ldots + P^n$ is a null matrix. Find the inverse of P and the eigenvalues of P.

[6+6]=12

4. (a) Suppose that there are five pairs of shoes in a closet and four shoes are taken out at random. What is the probability that, among the four which are taken out, there is at least one complete pair?

(b) Two identical independent components having lifetime T_1 and T_2, respectively, are connected in a parallel system. Suppose that the distributions of both T_1 and T_2 are exponential initially with mean $1/\lambda$. But, whenever one component fails, the lifetime distribution of the remaining component changes to exponential with mean $1/\alpha$. If T denotes the overall lifetime of the system, find $P(T \geq t)$ for any $t > 0$.

[5+7]=12

5. (a) Suppose that (X, Y) has a joint distribution with $E(Y|X = x) = x^3$ for all $x \in \mathbb{R}$. If the marginal distribution of X is $\mathcal{N}(0, 1)$, then prove that

$$Correlation(X, Y) > 0.$$

(b) Suppose that $(X, Y) \sim BN(0, 0, 1, 1, \rho)$, and define

$$Z = \frac{X - Y}{X + Y}\sqrt{\frac{1 + \rho}{1 - \rho}}.$$

Show that the distribution of Z is symmetric, and find $P(Z < 0|X + Y < 0)$.

[5+3+4]=12

6. Let $X_1, \ldots, X_n$ be independent and identically distributed as $\text{Bin}(m, p)$, where both m and p are unknown.

a) Write down the first two method of moments equations and solve them to find the estimators $\widehat{m}$ and $\widehat{p}$.

b) Show that $\widehat{m}$ and $\widehat{p}$ are consistent for m and p, respectively.

c) Show that $\widehat{m}$ and $\widehat{p}$ are jointly asymptotically normal in the sense that

$$\sqrt{n}\begin{pmatrix} \widehat{m} - m \\ \widehat{p} - p \end{pmatrix} \Rightarrow \mathcal{N}_2(0, \Sigma),$$

where Σ is a covariance matrix. Find Σ.

[3+3+6]=12

7. Consider the following linear model:

$$\begin{aligned}
Y_1 &= \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \epsilon_1 \\
Y_2 &= \theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 - \theta_6 + \epsilon_2 \\
Y_3 &= \theta_1 - \theta_2 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + \epsilon_3 \\
Y_4 &= \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5 + \theta_6 + \epsilon_4 \\
Y_5 &= -\theta_1 + \theta_2 - \theta_3 + \theta_4 - \theta_5 + \theta_6 + \epsilon_5 \\
Y_6 &= -\theta_1 + \theta_2 - \theta_3 - \theta_4 + \theta_5 - \theta_6 + \epsilon_6
\end{aligned}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_6)'$, with $E(\boldsymbol{\epsilon}) = \mathbf{0}$, and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbb{I}$.

a) Find an unbiased estimator of θ_i for all $i = 1, \ldots, 6$.

b) Find the Best Linear Unbiased Estimator (BLUE) of $\theta_1 - 2\theta_2 + \theta_3$.

[7+5]=12

8. Consider a linear model

$$y_i = \alpha + \beta x_i + \gamma z_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where independent observations $\{y_i\}$ of a response variable Y are regressed on two regressors X and Z and $\{\epsilon_i\}$ are unobserved independent and identically distributed with $E(\epsilon_i) = 0$ and $Var(\epsilon_i) < \infty$ for all i.

a) Write down the normal equation for estimating the parameters (α, β, γ) by ordinary least square method. Let us denote the resulting estimates by $(\widehat{\alpha}_{OLS}, \widehat{\beta}_{OLS}, \widehat{\gamma}_{OLS})$.

b) Suppose we alternatively fit the model as follows. First we regress Y on X by minimizing $\sum_i (y_i - \alpha_1 - \beta_1 x_i)^2$ with respect to (α_1, β_1) to get their estimates as $(\widehat{\alpha}_1, \widehat{\beta}_1)$ and compute the residuals $e_i = y_i - \widehat{\alpha}_1 - \widehat{\beta}_1 x_i$ for each i. In the next step, we regress $\{e_i\}$ on Z by minimizing the criterion $\sum_i (e_i - \alpha_2 - \gamma_2 z_i)^2$ with respect to (α_2, γ_2) to get their estimates as $(\widehat{\alpha}_2, \widehat{\gamma}_2)$. Show that $\widehat{\alpha}_2 = 0$ and $\widehat{\gamma}_2 = \widehat{\gamma}_{OLS}$.

[3+9]=12

9. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function F_θ. Suppose there exists a positive integer m such that $g(X_1, \ldots, X_m)$ is unbiased for θ and $E[g(X_1, \ldots, X_m)^2] < \infty$. Prove that if there exists a UMVUE of θ for any $n > m$, the variance of this UMVUE must converge to zero as $n \to \infty$.

[12]

10. Suppose that a sample of size n is drawn using SRSWR from a finite population of N units, where $N > n$ and $N \geqslant 3$. Let $\bar{y}$ be the sample mean of the study variables corresponding to the n selected units. Now, let us assume that one variate value y_1 corresponding to one unit is known and consequently a simple random sample of size n without replacement are now drawn from the remaining $(N-1)$ units; denote the sample mean of the study variables corresponding to these n selected units by $\bar{y}_0$. Consider the following two estimators for the population total as given by

$$t_1 = N\bar{y}, \quad \text{and} \quad t_2 = (N-1)\bar{y}_0 + y_1.$$

Prove that

a) t_2 is unbiased for the population total.

b) $Var(t_1) \geq Var(t_2)$.

[5+7]=12